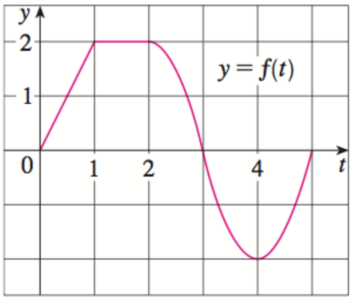
LECTURE: 5-3 THE FUNDAMENTAL THEOREM OF CALCULUS (PART 1)

Example 1: If f is the function whose graph is shown and $g(x) = \int_0^x f(t)dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$ and $g(5)$. Then, sketch a rough graph of g.



$g(0) = \int_0^0 f(t)\,dt = 0$

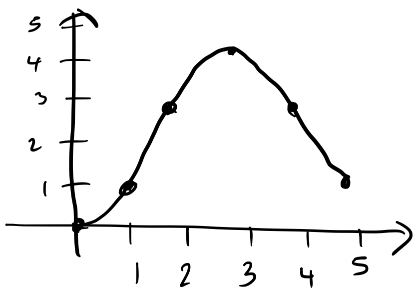
$g(1) = \int_0^1 f(t)\,dt = \frac{1}{2}(1)(2) = 1$

$g(2) = \int_0^2 f(t)\,dt = 1 + 2 = 3$

$g(3) = \int_0^3 f(t)\,dt \approx 3 + 1.5 = 4.5$

$g(4) = \int_0^4 f(t)\,dt = 4.5 - 1.5 = 3$

$g(5) = \int_0^5 f(t)\,dt \approx 3 - 1.5 = 1.5$



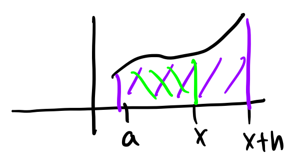
> **The Fundamental Theorem of Calculus, Part 1** If f is continuous on $[a, b]$, the function g defined by
>
> $$g(x) = \int_a^x f(t)dt \quad a \le x \le b$$
>
> is continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) = f(x)$.

① def of a derivative:

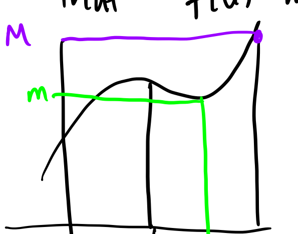
$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \right)$$

$$= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt$$



② Since f is continuous on $[a,b]$, it is cts on $[x, x+h]$ by Extreme Value Thm, there are u, v in $(x, x+h)$ such that $f(u) = m$ and $f(v) = M$



and $\longrightarrow mh \le \int_x^{x+h} f(t)\,dt \le Mh$

$$f(u) \le \frac{1}{h} \int_x^{x+h} f(t)\,dt \le f(v)$$

Since $\lim_{h\to 0} f(u) = f(x)$ & $\lim_{h\to 0} f(v) = f(x)$

Thus $g'(x) \to f(x)$

Example 2: The Fresnel function $S(x) = \int_0^x \sin(\pi t^2/2)dt$ first appeared in Fresnel's theory of the diffraction of light waves. Recently it was be applied to the design of highways. Find the derivative of the Fresnel function.

$$S'(x) = \frac{d}{dx}\int_0^x \sin(\pi t^2/2)\,dt$$
$$= \boxed{\sin(\pi x^2/2)}$$

Example 3: Find the derivative of the following functions.

(a) $g(x) = \int_1^{x^4} \sec t dt$

$$g'(x) = \frac{d}{dx}\int_1^{x^4} \sec t\,dt$$
$$= \sec(x^4)\cdot \frac{d}{dx} x^4$$
$$= \boxed{4x^3 \sec(x^4)}$$

(b) $g(x) = \int_{2x+1}^{2} \sqrt{t}dt$ $= -\int_2^{2x+1} \sqrt{t}\,dt$

$$g'(x) = \frac{d}{dx}\left(-\int_2^{2x+1}\sqrt{t}\,dt\right)$$
$$= -\sqrt{2x+1}\cdot\frac{d}{dx}(2x+1)$$
$$= \boxed{-2\sqrt{2x+1}}$$

Example 4: Find the derivative of $g(x) = \int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}dt$

$$g(x) = \int_{\tan x}^{a} \frac{1}{\sqrt{2+t^4}}\,dt + \int_a^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt$$
$$= -\int_a^{\tan x} \frac{1}{\sqrt{2+t^4}}\,dt + \int_a^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt$$

$$g'(x) = \frac{d}{dx}\left(-\int_a^{\tan x}\frac{1}{\sqrt{2+t^4}}\,dt\right) + \frac{d}{dx}\int_a^{x^2}\frac{1}{\sqrt{2+t^4}}\,dt$$
$$= -\frac{1}{\sqrt{2+\tan^4 x}}\frac{d}{dx}\tan x + \frac{1}{\sqrt{2+(x^2)^4}}\frac{d}{dx}x^2$$
$$= \boxed{-\frac{\sec^2 x}{\sqrt{2+\tan^4 x}} + \frac{2x}{\sqrt{2+x^8}}}$$

The Fundamental Theorem of Calculus (Part 2) If f is continuous on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any anti derivative of f, that is, is a function such that $F' = f$

Let $g(x) = \int_a^x f(x)\,dx$. We know $g'(x) = f(x)$; that is g is the anti-derivative of f.

If F is any other anti derivative we know F & g differ by a constant: $F(x) = g(x) + C$ for $a < x < b$

Now: $F(b) - F(a) = g(b) + C - (g(a) + C)$

$$= g(b) - g(a)$$

$$= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt$$

$$= \int_a^b f(t)\,dt.$$

Example 5: Evaluate the following integrals.

a) $\displaystyle\int_0^1 x^2 dx = \frac{1}{3}x^3 \Big|_0^1$

$$= \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3$$

$$= \boxed{1/3}$$

b) $\displaystyle\int_0^4 (1 + 3y - y^2)dy = \left(y - \frac{3}{2}y^2 - \frac{1}{3}y^3\right)\Big|_0^4$

$$= \left(4 - \frac{3}{2}(16) - \frac{1}{3}(64)\right) - (0)$$

$$= 4 - 24 - \frac{64}{3}$$

$$= -20\left(\frac{3}{3}\right) - \frac{64}{3}$$

$$= \boxed{\frac{-124}{3}}$$

To compute integrals effectively you **must** have your basic anti-differentiation formulas down. You should know that anti-derivatives to the following functions. Note, I'm going to use the $\int$ symbol to mean "find the anti-derivative" of the function right after the symbol.

Anti-Derivatives of Common Functions:

- $\int x^n dx = x^{n+1}/(n+1)$
- $\int \sin x dx = -\cos x$
- $\int \cos x dx = \sin x$
- $\int \sec^2 x dx = \tan x$
- $\int \sec x \tan x dx = \sec x$
- $\int \csc^2 x dx = -\cot x$
- $\int \csc x \cot x dx = -\csc x$
- $\int e^x dx = e^x$
- $\int a^x dx = \frac{a^x}{\ln a}$
- $\int \frac{1}{1+x^2} dx = \tan^{-1}(x)$
- $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x)$
- $\int \frac{1}{x} du = \ln|x|$

Example 6: Evaluate the following integrals.

(a) $\int_2^5 \frac{3}{x} dx = 3\ln|x| \Big|_2^5$

$= 3\ln 5 - 3\ln 2$

$= \boxed{3(\ln 5 - \ln 2)}$

$= \boxed{3\ln\left(\frac{5}{2}\right)}$

(b) $\int_0^{\pi/2} \cos x dx = \sin x \Big|_0^{\pi/2}$

$= \sin(\pi/2) - \sin(0)$

$= 1 - 0$

$= \boxed{1}$

Example 7: Evaluate the following integrals.

(a) $\int_1^8 \sqrt[3]{x} dx = \int_1^8 x^{1/3} dx$

$= \frac{3}{4} x^{4/3} \Big|_1^8$

$= \frac{3}{4}(8^{4/3} - 1^{4/3})$

$= \frac{3}{4}(\sqrt[3]{8}^4 - 1)$

$= \frac{3}{4}(16-1)$

$= \boxed{45/4}$

(b) $\int_{\pi/6}^{\pi/2} \csc x \cot x dx$

$= -\csc x \Big|_{\pi/6}^{\pi/2}$

$= \frac{-1}{\sin \pi/2} + \frac{1}{\sin \pi/6}$

$= -1 + 1/(1/2)$

$= -1 + 2$

$= \boxed{1}$

(c) $\int_0^1 \frac{9}{1+x^2} dx = 9\tan^{-1} x \Big|_0^1$

$= 9(\tan^{-1} 1 - \tan^{-1} 0)$

$= 9(\pi/4 - 0)$

$= \boxed{\frac{9\pi}{4}}$

Example 8: We do not have any product or quotient rules for anti-differentiation. To evaluate an integral that is expressed as a product or quotient you must try to manipulate the integrand (the stuff inside the $\int$ sign) to look like something you know how to anti-differentiate. The following integrals are examples of this. Evaluate the following integrals.

(a) $\displaystyle\int_1^3 \frac{x^3+3x^6}{x^4}dx = \int_1^3 \frac{x^3}{x^4} + \frac{3x^6}{x^4}\,dx$

$= \displaystyle\int_1^3 \left(\frac{1}{x} + 3x^2\right)dx$

$= \left(\ln|x| + x^3\right)\Big|_1^3$

$= \ln 3 + 27 - (\ln 1 + 1)$

$= \boxed{\ln 3 + 26}$

(b) $\displaystyle\int_0^1 x(3+\sqrt{x})dx = \int_0^1 \left(3x + x\cdot x^{1/2}\right)dx$

$= \displaystyle\int_0^1 \left(3x + x^{3/2}\right)dx$

$= \left(\frac{3}{2}x^2 + \frac{2}{5}x^{5/2}\right)\Big|_0^1$

$= \frac{3}{2}\frac{5}{5} + \frac{2}{5}\frac{2}{2}$

$= \frac{15}{10} + \frac{4}{10} = \boxed{19/10}$

Example 9: Evaluate the following integrals.

(a) $\displaystyle\int_0^2 (5^x + x^5)dx = \left(\frac{5^x}{\ln 5} + \frac{x^6}{6}\right)\Big|_0^2$

$= \frac{5^2}{\ln 5} + \frac{2^6}{6} - \left(\frac{5^0}{\ln 5} + 0\right)$

$= \frac{25}{\ln 5} - \frac{64}{6} - \frac{1}{\ln 5}$

$= \boxed{24/\ln 5 - 32/3}$

(b) $\displaystyle\int_{1/2}^{\sqrt{2}/2} \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x \Big|_{1/2}^{\sqrt{2}/2}$

$= \sin^{-1}(\sqrt{2}/2) - \sin^{-1}(1/2)$

$= \frac{3}{3}\pi/4 - \pi/6\frac{2}{2}$

$= \frac{3\pi}{12} - \frac{2\pi}{12}$

$= \boxed{\pi/12}$

Example 10: What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}dx = \frac{x^{-1}}{-1}\Bigg]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

The function $f(x) = 1/x^2$ is not continuous on $[-1, 3]$, and the FTC #2 does not apply.